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On the point spectrum of the linear gas model

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Abstract. The discrete spectrum of the Rayleigh piston is investigated using a WKB method for integral operators. Analytic formulae asymptotically valid for eigenvalues and eigenfunctions are obtained. The discrete spectrum is finite for $0 < \gamma \ll 1$.

1. Introduction

The one-dimensional test particle gas problem, or Rayleigh piston, has come into focus again in recent years (Hoare 1971, Hoare and Rahman 1973, 1974, Barker *et al* 1977, 1981).

The interest in this model comes from its generality, i.e. it may be viewed as a prototype for all linear statistical dynamical models which are sufficiently simple, retain some vestiges of reality and embody all subtleties of the mathematical problem.

One considers an ensemble of frictionless 'pistons' of mass M undergoing interactions with a one-dimensional heat bath of particles of mass m and temperature T. Mathematically, the model is described by a singular master equation which takes the form of a multiplication operator perturbed by a Hilbert–Schmidt integral operator (Hoare 1971). The spectrum of this operator consists of a continuum region to which one adds, in some cases, a finite number of eigenvalues. Although there have been some attempts to diagonalise this operator (Hoare and Rahman 1973, 1974, Barker *et al* 1977) the problem has not been solved until now. A precise estimation of the first eigenvalues is of great interest, the long-time behaviour of the system being dominated by them. A numerical treatment of the discrete eigenvalue problem was given by Barker *et al* (1981).

The aim of this paper is to obtain analytic estimates upon eigenvalues and eigenfunctions for the case when the parameter $\gamma = m/M$ is small, $0 \leqslant \gamma < 1$.

2. Rayleigh limit

The master equation has the form (Barker *et al* 1981)

$$\frac{\partial Q(x,t)}{\partial t} = A\gamma Q(x,t) = \int_{-\infty}^{\infty} A(y,x)Q(y,t)\,\mathrm{d}y \tag{1}$$

in which $Q(x,t)$ is the probability density for test particles at velocity x and $A\gamma$ is a

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singular integral operator having the kernel $A(y, x) = K(y, x) - z(x)\delta(x-y)$ with

$$K(y, x) = \mu^2|x-y| \exp\{-[(y-x)\mu + x]^2\}$$

$$z(x) = \int_{-\infty}^{\infty} K(x, y)\, dy = {}_1F_1(-\tfrac{1}{2}, \tfrac{1}{2}; -x^2)$$

$$\mu^{-1} = 2\gamma/(1+\gamma).$$

${}_1F_1(a, b; x)$ denotes the confluent hypergeometric function.

By using the transformation $Q(x, t) = \exp(x^2/2\gamma)P(x, t)$ followed by a new scaling of the independent variable, $x = \mu^{-1}u$, (1) takes the form

$$\frac{\partial P(u, t)}{\partial t} + z(\mu^{-1}u)P(u, t) = \int_{-\infty}^{\infty} |u-v| \exp[-\alpha(u^2+v^2) - 2\beta uv]P(v, t)\, dv \qquad (1a)$$

where

$$\alpha = (\gamma^2+1)/(1+\gamma)^2 \qquad \text{and} \qquad \beta = (\gamma^2-1)/(1+\gamma)^2.$$

If we look for solutions of the form $P(u, t) = \exp(-\lambda t)f(u)$, $(1a)$ becomes the eigenvalue equation

$$(z(\mu^{-1}u) - \lambda)f(u) = \int_{-\infty}^{\infty} |u-v| \exp[-\alpha(u^2+v^2) - 2\beta uv]f(v)\, dv. \qquad (1b)$$

The relations $(1a)$ and $(1b)$ are important since we can take the limit $\gamma \to 0$ and still obtain well defined operators. In the limit $\gamma \to 0$ $(1b)$ becomes

$$(z(0) - \lambda)f(u) = \int_{-\infty}^{\infty} |u-v| \exp[-(u-v)^2]f(v)\, dv \qquad (2)$$

which is a convolution equation. Since its kernel is an even function, the orthonormal eigenfunctions are

$$f_p(u) = \pi^{-1/2} \cos pu \qquad (3)$$

and they are indexed by a continuous parameter p, $p \geq 0$. Hence the spectrum of the operator is $[0, \infty)$, and in this way we have obtained a rigorous proof of a result which was conjectured some time ago (Barker *et al* 1981) but, to our knowledge, never proved.

By the substitution of (3) into (2), we obtain $\lambda = 1 - {}_1F_1(1, \tfrac{1}{2}; -\tfrac{1}{4}p^2)$ so that $P(u, t)$ has the form

$$P(u, t) = \pi^{-1/2} e^{-t} \int_0^{\infty} \cos pu \exp[{}_1F_1(1, \tfrac{1}{2}; -\tfrac{1}{4}p^2)t]\, dp$$

and is singular for $u = 0$.

For $\gamma \neq 0$ the right-hand side of $(1b)$ is a Hilbert–Schmidt operator, and we shall consider it as a perturbation to the multiplication operator $z(\mu^{-1}u)$. Since the spectrum of the multiplication operator is absolutely continuous ($z(x)$ is a continuous function) it will remain unchanged when is perturbed by a completely continuous operator; the point spectrum may change (Birman and Solomjak 1980). So, for $\gamma \neq 0$, the operator spectrum is composed from a continuous part $[1, \infty)$ and, possibly, a finite number of discrete eigenvalues.

3. Discrete spectrum

We shall investigate now the discrete eigenvalues and eigenfunctions in the Rayleigh regime, $0<\gamma\ll 1$, and for this we shall use the WKB method in the form developed by Sirovich and Knight (1981) for integral operators.

Any integral kernel on the full line can be reexpressed as $K(x, y)=K[x-y, x+y]$, and the method gives asymptotic results for eigenvalues and eigenfunctions of kernels whose dependence upon $(x+y)$ is slow. If $\tilde{K}(p, q)$ is the Wigner transform of the original kernel

$$\tilde{K}(p, q)=\int_{-\infty}^{\infty} K(q+\tfrac{1}{2}u, q-\tfrac{1}{2}u) \exp(-\mathrm{i}up)\,\mathrm{d}u$$

λ_n will be asymptotically an eigenvalue if the closed curve $\tilde{K}(p, q)=\lambda_n$ encloses area $A(\lambda_n)=(2n+1)\pi$, $n=1,2,\ldots$.

Our kernel, in the limit $\gamma\to 0$, becomes a convolution kernel so its dependence upon $(x+y)$ is slow for $\gamma\ll 1$. A straightforward calculation shows that

$$\tilde{K}(p, q)=z(\mu^{-1}q)-(1+\gamma)^2 \exp(-\mu^{-2}q^2)\times {}_1F_1(1,\tfrac{1}{2}; -(1+\gamma)^2\tfrac{1}{4}p^2). \qquad (4)$$

Since it seems to be difficult inverting the relation $\tilde{K}(p, q)=\lambda_n$, we have made a Taylor expansion in (4) retaining terms up to second order only. Thus we found

$$\lambda_n=-\gamma(\gamma+2)+(2n+1)\gamma(\gamma+1)(2\gamma^2+4\gamma+4)^{1/2}, \qquad n=1,2,\ldots. \qquad (5)$$

If we retain only the leading term in (5) it takes the simpler form

$$\lambda_n=4n\gamma+\mathrm{O}(\gamma^2), \qquad n=1,2,\ldots. \qquad (5a)$$

In the same approximation the eigenfunctions are (Sirovich and Knight 1981)

$$f_n(u)=a_n \exp(-2\gamma u^2)H_n(2\gamma^{1/2}u) \qquad (6)$$

where $H_n(x)$ are Hermite polynomials and a_n normalisation constants.

Formula (5a) provides us with an explanation to numerical results found by Barker *et al* (1981), showing that up to higher-order corrections in γ, $\lambda_n/4\gamma$ are well approximated by integers. Formula (5) shows that the first discrete eigenvalue appears for $\gamma<0.17$ value close enough to numerical value $\gamma<0.28$ found by the same authors.

By a suitable choosing of the constants a_n (6) goes to (3) in the limit $\gamma\to 0$. Indeed if we choose $a_{2m}=(-1)^m m^{1/2}/2^{2m}m!$ and take the limits $\gamma\to 0$, $m\to\infty$ with $\gamma m=\frac{1}{16}p^2$ kept fixed, then

$$\lim_{m\to\infty} f_{2m}(u)=\lim_{m\to\infty} \exp(-p^2u^2/8m)a_{2m}H_{2m}(pu/2m^{1/2})=\pi^{-1/2}\cos pu$$

a result which confirms the consistency of our approach.

4. Conclusion

It is, of course, desirable to find a method for diagonalising the operator entering (1b). In this sense the results obtained by Hoare and Rahman (1973, 1974) and Barker *et al* (1977) are encouraging. Their results suggest that it would possible to find a low-order differential operator that commutes with the integral operator (1b). Since $z(x)$ is an even (double-valued on R) function the spectral multiplicity of our operator is equal to two, so that if such a differential operator does exist it will be of fourth order at least (Achiezer and Glazman 1978). In this sense the second-order differential operator obtained for the case $\gamma = 1$ seems to be a pure accident, being peculiar to this case. The possible existence of such a differential operator is under study.

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